Two New Applications of the Modified Extended Tanh-Function Method

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Based on a modified extended tanh-function method and symbolic computation, new exact solutions are found for a soliton breaking equation and coupled kdv system. The obtained solutions include rational, soliton, singular and periodical solutions.

Key words: Nonlinear Evolution Equation; Traveling Wave Solution; Symbolic Computation.

1. Introduction

Recently, we have introduced the *modified extended tanh function* (METF) method [1] to obtain multiple travelling wave solutions for nonlinear partial differential equations (PDEs). In this paper, using the METF method, we investigate the existance of other exact solutions for two nonlinear physical models. The first one is the soliton breaking equation

$$u_{xt} - 4u_x u_{xy} - 2u_y u_{xx} - u_{xxxy} = 0, \tag{1}$$

for which Fan et al. [2] found a kink-type and a singular solitary wave solutions. The second one is the coupled KdV system of equations

$$\begin{aligned} u_t &= -\alpha u_{xxx} + 6vv_x - 6\alpha u u_x, \\ v_t &= -\alpha v_{xxx} - 3\alpha u v_x, \end{aligned} \tag{2}$$

for which Wang et al. [3] obtained bell-type solitary wave solutions for u and v. Later, Fan et al. [2] found a bell-type solitary wave solution for u and a kink-type solitary wave solution for v.

Now, let us recall the main steps of the METF method. Consider a given PDE in two variables,

$$H(u, u_x, u_t, u_{xx}, \ldots) = 0. \tag{3}$$

We first consider its travelling solutions $u(x,t) = u(\zeta)$, $\zeta = x - \lambda t$. Then (3) becomes an ordinary differential equation. The next crucial step is that the solution we are looking for is expressed in the form

$$u(\zeta) = \sum_{i=0}^{m} a_i \omega^i + \sum_{i=0}^{m} b_i \omega^{-i}, \tag{4}$$

and

$$\omega' = k + \omega^2, \tag{5}$$

where k is a parameter to be determined, $\omega = \omega(\zeta)$, $\omega' = \frac{d\omega}{d\zeta}$. The parameter m can be found by balancing the highest-order linear term with the nonlinear terms [4 – 10]. Substituting (4) and (5) into the relevant ordinary differential equation will yield a system of algebraic equations with respect to a_i, b_i, k, and λ (where $i = 0, \ldots, m$) because all the coefficients of ω^i have to vanish. With the aid of *Mathematica*, one can determine a_i, b_i, k, and λ. The Riccati Equation (5) has the general solutions

$$\omega = \begin{cases} -\sqrt{-k}\tanh\sqrt{-k}\zeta, & \text{with } k < 0, \\ -\sqrt{-k}\coth\sqrt{-k}\zeta, & \text{with } k < 0, \end{cases} \tag{6}$$

$$\omega = -\frac{1}{\zeta}, \text{ with } k = 0, \tag{7}$$

and

$$\omega = \begin{cases} \sqrt{k}\tan\sqrt{k}\zeta, & \text{with } k > 0, \\ -\sqrt{k}\cot\sqrt{k}\zeta, & \text{with } k > 0. \end{cases} \tag{8}$$

2. The Soliton Breaking Equation

Introducing the transformation $u(x,y,t) = u(\zeta)$, where $\zeta = x+\alpha y-\lambda t$ into (1) leads to the ordinary differential equation

$$\alpha u'''' + (6\alpha u' + \lambda)u'' = 0. \tag{9}$$

Balancing the highest-order linear terms and nonlinear terms leads to

$$u(\zeta) = \sum_{i=0}^{1} a_i \omega^i + \sum_{i=0}^{1} b_i \omega^{-i}. \tag{10}$$

Substituting (10) into (9) and making use of (5), with the help of *Mathematica* we get a system of algebraic equations for a_0, a_1, b_0, b_1, α, k, and λ:

$$ka_1(8k\alpha+\lambda+6k\alpha a_1-6\alpha b_1) = 0,$$

$$a_1(20k\alpha+\lambda+12k\alpha a_1-6\alpha b_1) = 0,$$

$$\alpha a_1(2+a_1) = 0,$$

$$kb_1(8k\alpha+\lambda+6k\alpha a_1-6\alpha b_1) = 0,$$

$$k^2 b_1(20k\alpha+\lambda+6k\alpha a_1-12\alpha b_1) = 0,$$

and

$$k^3\alpha b_1(-2k+b_1) = 0.$$

From this, we find

$$k = 0,\ a_1 = -2,\ b_1 = \lambda/6\alpha, \tag{11}$$

$$\lambda = 4k\alpha,\ a_1 = -2,\ b_1 = 0, \tag{12}$$

$$\lambda = 4k\alpha,\ a_1 = 0,\ b_1 = 2k, \tag{13}$$

and

$$\lambda = 16k\alpha,\ a_1 = -2,\ b_1 = 2k. \tag{14}$$

According to (11), the solution to (1) reads

$$u(x,y,t) = (a_0+b_0) - \frac{\lambda}{6\alpha}\zeta + \frac{2}{\zeta} \quad \text{with } \zeta = x+\alpha y-\lambda t, \tag{15}$$

where λ, α, a_0, and b_0 are arbitrary constants.

Due to (12), for $k<0$ the solution to (1) reads

$$u(x,y,t) = (a_0+b_0) + 2\sqrt{-k}\tanh[\sqrt{-k}\zeta], \quad \text{where } \zeta = x+\alpha y-4k\alpha t, \tag{16}$$

while for $k>0$ it is

$$u(x,y,t) = (a_0+b_0) - 2\sqrt{k}\tan[\sqrt{k}\zeta], \quad \text{where } \zeta = x+\alpha y-4k\alpha t, \tag{17}$$

where α, a_0, and b_0 are arbitrary constants.

From (13) it is clear that for the case $k<0$ we get

$$u(x,y,t) = (a_0+b_0) + 2\sqrt{-k}\coth[\sqrt{-k}\zeta], \quad \text{where } \zeta = x+\alpha y-4k\alpha t, \tag{18}$$

while for $k>0$ it is

$$u(x,y,t) = (a_0+b_0) + 2\sqrt{k}\cot[\sqrt{k}\zeta], \quad \text{where } \zeta = x+\alpha y-4k\alpha t. \tag{19}$$

Finally, (14) leads for $k<0$ to

$$u(x,y,t) = (a_0+b_0) + 2\sqrt{-k}\{\coth[\sqrt{-k}\zeta] + \tanh[\sqrt{-k}\zeta]\}, \tag{20}$$

$$\text{where } \zeta = x+\alpha y-16k\alpha t,$$

and

$$u(x,y,t) = (a_0+b_0) + 2\sqrt{k}\{\cot[\sqrt{k}\zeta] - \tan[\sqrt{k}\zeta]\},$$

$$\text{where } \zeta = x+\alpha y-16k\alpha t, \tag{21}$$

for $k>0$, where α, a_0, and b_0 are arbitrary constants.

Solution (15) is a rational type solitary wave solution, while solution (16) is a kink-type one. Since a_0, b_0, k, and α are left arbitrary, we could make the transformation $(a_0+b_0) \to a$, $\alpha \to \frac{1}{-4k}$, and $k \to -k_0^2$. Then the solution (16) equals exactly that in [2] (The kink-type soliton solution of (1) given in [2] contains some minor errors. The correct form is $u(x,t) = a+2k_0\tanh[k_0(x+y/4k_0^2+t)]$). Solutions (17–21) are triangle-type periodical solutions.

3. The Coupled KdV Equations

Introducing the transformation $u(x,t) = u(\zeta)$, where $\zeta = x-\lambda t$, into (2) leads to the coupled system of ordinary differential equations

$$\alpha u''' - 6vv' + (6\alpha u-\lambda)u' = 0, \quad \alpha v''' + (3\alpha u-\lambda)v' = 0. \tag{22}$$

Balancing the highest-order linear terms and nonlinear terms leads to

$$u(\zeta)=\sum_{i=0}^{2}a_i\omega^i+\sum_{i=0}^{2}b_i\omega^{-i},\qquad v(\zeta)=\sum_{i=0}^{2}s_i\omega^i+\sum_{i=0}^{2}r_i\omega^{-i}. \tag{23}$$

Substituting (23) into (22) and making use of (5), with the help of *Mathematica* we get a system of algebraic equations, for a_0, a_1, a_2, b_0, b_1, b_2, s_0, s_1, s_2, r_0, r_1, r_2, k, and λ:

$$k[3\alpha a_1^2+a_2(8k\alpha-\lambda+6\alpha a_0+6\alpha b_0)-3s_1^2-6r_0s_2-6s_0s_2]=0,$$
$$a_1[8k\alpha-\lambda+6\alpha(a_0+3ka_2+b_0)]-6[-\alpha a_2b_1+r_1s_2+s_1(r_0+s_0+3ks_2)]=0,$$
$$3\alpha a_1^2+a_2[20k\alpha-\lambda+6\alpha(a_0+b_0)]-6s_2(r_0+s_0+ks_2)+6k\alpha a_2^2-3s_1^2=0,$$
$$\alpha a_1(1+3a_2)-3s_1s_2=0,$$
$$\alpha a_2(2+a_2)-s_2^2=0,$$
$$3\alpha b_1^2+b_2[8k\alpha-\lambda+6\alpha(a_0+b_0)]-3(r_1^2+2r_2(r_0+s_0))=0,$$
$$-b_1[k(8k\alpha-\lambda+6\alpha(a_0+b_0))+18\alpha b_2]+6[kr_2s_1-k\alpha a_1b_2+r_1(3r_2+k(r_0+s_0))]=0,$$
$$3k\alpha b_1^2+kb_2[20k\alpha-\lambda+6\alpha(a_0+b_0)]+6\alpha b_2^2-3[kr_1^2+2r_2(r_2+k(r_0+s_0))]=0,$$
$$k[\alpha b_1(k^2+3b_2)-3r_1r_2]=0,$$
$$k[r_2^2-\alpha b_2(2k^2+b_2)]=0,$$
$$b_1[\lambda-2\alpha-6\alpha(a_0-ka_2+b_0)]+a_1k[2k\alpha-\lambda+6\alpha(a_0+b_0)]-6s_1[-r_2+k(r_0+s_0)]-6\alpha a_1b_2-6kr_1s_2+6r_1(r_0+s_0)=0,$$
$$3\alpha[-a_1r_1-2a_2r_2+s_1(ka_1+b_1)]+2s_2[3\alpha b_2+k(8k\alpha-\lambda+3\alpha(a_0+b_0))]=0,$$
$$s_1[8k\alpha-\lambda+3\alpha(a_0+b_0)]-3\alpha a_2(r_1-ks_1)+6\alpha s_2(ka_1+b_1)=0,$$
$$3\alpha a_1s_1+2s_2[20k\alpha-\lambda+3\alpha(a_0+ka_2+b_0)]=0,$$
$$\alpha[2a_1s_2+s_1(2+a_2)]=0,$$
$$\alpha s_2(4+a_2)=0,$$
$$6k\alpha b_2s_2-3k\alpha a_1r_1-3\alpha b_1(r_1-ks_1)-2r_2[8k\alpha-\lambda+3\alpha(a_0+ka_2+b_0)]=0,$$
$$3k\alpha b_2s_1-6\alpha r_2(ka_1+b_1)-r_1[3\alpha b_2+k(8k\alpha-\lambda+3\alpha(a_0+b_0))]=0,$$
$$3k\alpha b_1r_1+2r[3\alpha b_2+k(20k\alpha-\lambda+3\alpha(a_0+b_0))]=0,$$
$$k\alpha[2b_1r_2+r_1(b_2+2k^2)]=0,$$
$$k\alpha r_2(b_2+4k^2)=0,$$

and

$$6k\alpha b_1s_2-6\alpha a_1r_2+r_1[-2k\alpha+\lambda-3\alpha(a_0+ka_2+b_0)]+s_1[3\alpha b_2+k(2k\alpha-\lambda+3\alpha(a_0+b_0))]=0.$$

Solving the above system, with the aid of *Mathematica* we get eight different cases:

$$\lambda=\alpha(8k+3a_0+3b_0), a_1=0,\ a_2=-4,\ b_1=0,\ b_2=0,\ r_1=0,\ r_2=0,\ s_0=\mp\sqrt{\frac{\alpha}{2}}(a_0+b_0)-r_0,\ s_1=0,\ s_2=\pm2\sqrt{2\alpha},\ \text{where}\ \alpha\neq0, \tag{24}$$

$$\lambda=\alpha(8k+3a_0+3b_0),\ a_1=0,\ a_2=-4,\ b_1=0,\ b_2=-4k^2,\ r_1=0,\ r_2=\mp2\sqrt{2\alpha}k^2,\ s_0=\pm\sqrt{\frac{\alpha}{2}}(a_0+b_0)-r_0,\ s_1=0,\ s_2=\mp2\sqrt{2\alpha},\ \text{where}\ k\neq0\ \text{and}\ \alpha\neq0, \tag{25}$$

$$\lambda=\alpha(8k+3a_0+3b_0),\ a_1=0,\ a_2=0,\ b_1=0,\ b_2=-4k^2,\ r_1=0,\ r_2=\mp2\sqrt{2\alpha}k^2,\ s_0=\pm\sqrt{\frac{\alpha}{2}}(a_0+b_0)-r_0,\ s_1=0,\ s_2=0,\ \text{where}\ k\neq0\ \text{and}\ \alpha\neq0, \tag{26}$$

$$k=0,\ \lambda=-\frac{3}{2}s_1^2,\ a_0=-\frac{2\alpha b_0-s_1^2}{2\alpha},\ a_1=0,\ a_2=-2,\ b_1=0,\ b_2=0,\ r_1=0,\ r_2=0,\ s_0=-r_0,\ s_2=0,\ \text{where}\ s_1\neq0\ \text{and}\ \alpha\neq0, \tag{27}$$

$$\lambda=-16\alpha k-\frac{3r_1^2}{2k^2},\ a_0=-4k-b_0-\frac{r_1^2}{2\alpha k^2},\ a_1=0,$$
$$a_2=-2,\ b_1=0,\ b_2=-2k^2,\ r_2=0,\ s_0=-r_0,$$
$$s_1=-\frac{r_1}{k},\ s_2=0,\ \text{where } k\neq 0,\ \alpha\neq 0 \text{ and } r_1\neq 0, \quad (28)$$

$$\lambda=-4\alpha k-\frac{3r_1^2}{2k^2},\ a_0=-2k-b_0-\frac{r_1^2}{2\alpha k^2},\ a_1=0,$$
$$a_2=0,\ b_1=0,\ b_2=-2k^2,\ r_2=0,\ s_0=-r_0,$$
$$s_1=0,\ s_2=0,\ \text{where } k\neq 0,\ \alpha\neq 0 \text{ and } r_1\neq 0, \quad (29)$$
$$\lambda=8\alpha k-\frac{3r_1^2}{2k^2},\ a_0=-b_0-\frac{r_1^2}{2\alpha k^2},\ a_1=0,\ a_2=-2,$$
$$b_1=0,\ b_2=-2k^2,\ r_2=0,\ s_0=-r_0,\ s_1=\frac{r_1}{k},$$
$$s_2=0,\ \text{where } k\neq 0,\ \alpha\neq 0 \text{ and } r_1\neq 0, \quad (30)$$
and
$$\lambda=-4\alpha k-\frac{3}{2}s_1^2,\ a_0=-2k-b_0-\frac{s_1^2}{2\alpha},\ a_1=0,$$
$$a_2=-2,\ b_1=0,\ b_2=0,\ r_1=0,\ r_2=0,\ s_0=-r_0,$$
$$s_2=0,\ \text{where } k\neq 0,\ \alpha\neq 0 \text{ and } s_1\neq 0. \quad (31)$$

Hence we got the following set of solutions for (2):

(i) As $k<0$:
$$u(x,t)=(a_0+b_0)+4k\tanh^2\left[\sqrt{-k}(x-\lambda t)\right],$$
$$v(x,t)=\mp\sqrt{\frac{\alpha}{2}}(a_0+b_0)\mp 2\sqrt{2\alpha}k\tanh^2\left[\sqrt{-k}(x-\lambda t)\right],$$
where $\lambda=\alpha(8k+3a_0+3b_0)$ and $\alpha\neq 0$; (32)
$$u(x,t)=(a_0+b_0)+4k\tanh^2\left[\sqrt{-k}(x-\lambda t)\right]+4k\coth^2\left[\sqrt{-k}(x-\lambda t)\right],$$
$$v(x,t)=\pm\left(\sqrt{\frac{\alpha}{2}}(a_0+b_0)+2\sqrt{2\alpha}k\tanh^2\left[\sqrt{-k}(x-\lambda t)\right]+2\sqrt{2\alpha}k\coth^2\left[\sqrt{-k}(x-\lambda t)\right]\right),$$
where $\lambda=\alpha(8k+3a_0+3b_0)$ and $\alpha\neq 0$; (33)
$$u(x,t)=(a_0+b_0)+4k\coth^2\left[\sqrt{-k}(x-\lambda t)\right],$$
$$v(x,t)=\mp\sqrt{\frac{\alpha}{2}}(a_0+b_0)\pm 2\sqrt{2\alpha}k\coth^2\left[\sqrt{-k}(x-\lambda t)\right],$$
where $\lambda=\alpha(8k+3a_0+3b_0)$ and $\alpha\neq 0$; (34)
$$u(x,t)=-(4k+\frac{r_1^2}{2k^2\alpha})+2k\tanh^2\left[\sqrt{-k}(x-\lambda t)\right]+2k\coth^2\left[\sqrt{-k}(x-\lambda t)\right],$$
$$v(x,t)=\frac{r_1}{\sqrt{-k}}\left(\tanh\left[\sqrt{-k}(x-\lambda t)\right]-\coth\left[\sqrt{-k}(x-\lambda t)\right]\right),$$
where $\lambda=-16\alpha k-\frac{3r_1^2}{2k^2}$, $\alpha\neq 0$ and $r_1\neq 0$; (35)
$$u(x,t)=-(2k+\frac{r_1^2}{2k^2\alpha})+2k\coth^2\left[\sqrt{-k}(x-\lambda t)\right],$$
$$v(x,t)=-\frac{r_1}{\sqrt{-k}}\coth\left[\sqrt{-k}(x-\lambda t)\right],$$
where $\lambda=-4\alpha k-\frac{3r_1^2}{2k^2}$, $\alpha\neq 0$ and $r_1\neq 0$; (36)
$$u(x,t)=-\frac{r_1^2}{2k^2\alpha}+2k\tanh^2\left[\sqrt{-k}(x-\lambda t)\right]+2k\coth^2\left[\sqrt{-k}(x-\lambda t)\right],$$
$$v(x,t)=-\frac{r_1}{\sqrt{-k}}\left(\tanh\left[\sqrt{-k}(x-\lambda t)\right]+\coth\left[\sqrt{-k}(x-\lambda t)\right]\right),$$
where $\lambda=8\alpha k-\frac{3r_1^2}{2k^2}$, $\alpha\neq 0$ and $r_1\neq 0$; (37)
$$u(x,t)=-(2k+\frac{s_1^2}{2\alpha})+2k\tanh^2\left[\sqrt{-k}(x-\lambda t))\right],$$
$$v(x,t)=-\sqrt{-k}\tanh\left[\sqrt{-k}(x-\lambda t)\right],$$
where $\lambda=-4\alpha k-\frac{3s_1^2}{2}$, $\alpha\neq 0$ and $s_1\neq 0$. (38)

(ii) As $k>0$:
$$u(x,t)=(a_0+b_0)-4k\tan^2\left[\sqrt{k}(x-\lambda t)\right],$$
$$v(x,t)=\mp\sqrt{\frac{\alpha}{2}}(a_0+b_0)\pm 2\sqrt{2\alpha}k\tan^2\left[\sqrt{k}(x-\lambda t)\right],$$
where $\lambda=\alpha(8k+3a_0+3b_0)$ and $\alpha\neq 0$; (39)

$$u(x,t) = (a_0+b_0) - 4k\tan^2\left[\sqrt{k}(x-\lambda t)\right] - 4k\cot^2\left[\sqrt{k}(x-\lambda t)\right],$$

$$v(x,t) = \mp\left\{-\sqrt{\frac{\alpha}{2}}(a_0+b_0) + 2\sqrt{2\alpha}k\tan^2\left[\sqrt{k}(x-\lambda t)\right] + 2\sqrt{2\alpha}k\cot^2\left[\sqrt{k}(x-\lambda t)\right]\right\},$$

where $\lambda = \alpha(8k+3a_0+3b_0)$ and $\alpha \neq 0$; (40)

$$u(x,t) = (a_0+b_0) - 4k\cot^2\left[\sqrt{k}(x-\lambda t)\right],$$

$$v(x,t) = \pm\sqrt{\frac{\alpha}{2}}(a_0+b_0) \mp 2\sqrt{2\alpha}k\cot^2\left[\sqrt{k}(x-\lambda t)\right],$$

where $\lambda = \alpha(8k+3a_0+3b_0)$ and $\alpha \neq 0$; (41)

$$u(x,t) = -(4k+\frac{r_1^2}{2k^2\alpha}) - 2k\tan^2\left[\sqrt{k}(x-\lambda t)\right] - 2k\cot^2\left[\sqrt{k}(x-\lambda t)\right],$$

$$v(x,t) = \frac{r_1}{\sqrt{k}}\Big(\cot\left[\sqrt{k}(x-\lambda t)\right] - \tan[\sqrt{k}(x-\lambda t)]\Big),$$

where $\lambda = -16\alpha k - \frac{3r_1^2}{2k^2}$, $\alpha \neq 0$ and $r_1 \neq 0$; (42)

$$u(x,t) = -(2k+\frac{r_1^2}{2k^2\alpha}) - 2k\cot^2\left[\sqrt{k}(x-\lambda t)\right],$$

$$v(x,t) = \frac{r_1}{\sqrt{k}}\cot\left[\sqrt{k}(x-\lambda t)\right],$$

where $\lambda = -4\alpha k - \frac{3r_1^2}{2k^2}$, $\alpha \neq 0$ and $r_1 \neq 0$; (43)

$$u(x,t) = -\frac{r_1^2}{2k^2\alpha} - 2k\tan^2\left[\sqrt{k}(x-\lambda t)\right] - 2k\cot^2\left[\sqrt{k}(x-\lambda t)\right],$$

$$v(x,t) = \frac{r_1}{\sqrt{k}}\Big(\cot\left[\sqrt{k}(x-\lambda t)\right] + \tan\left[\sqrt{k}(x-\lambda t)\right]\Big),$$

where $\lambda = 8\alpha k - \frac{3r_1^2}{2k^2}$, $\alpha \neq 0$ and $r_1 \neq 0$; (44)

$$u(x,t) = -(2k+\frac{s_1^2}{2\alpha}) - 2k\tan^2\left[\sqrt{k}(x-\lambda t)\right],$$

$$v(x,t) = \sqrt{k}s_1\tan\left[\sqrt{k}(x-\lambda t)\right],$$

where $\lambda = -4\alpha k - \frac{3s_1^2}{2}$, $\alpha \neq 0$ and $s_1 \neq 0$. (45)

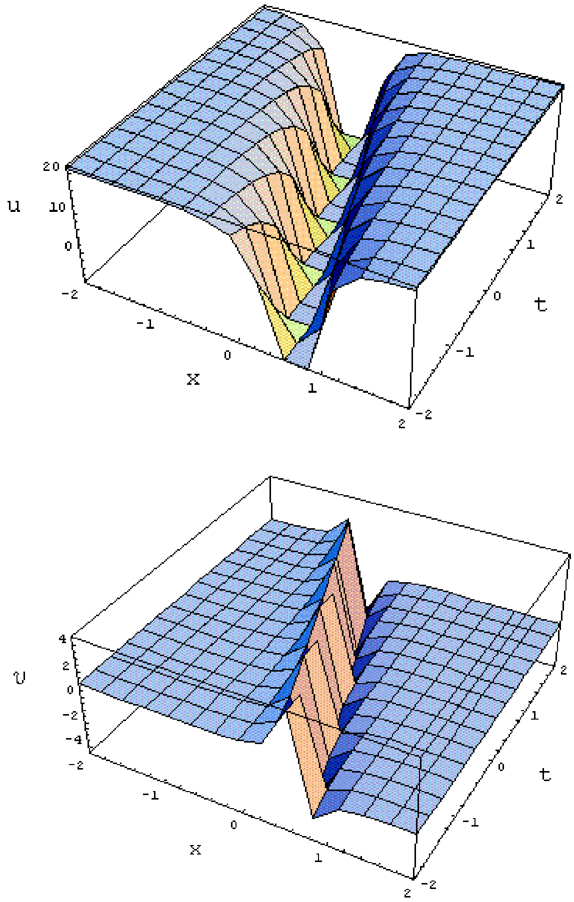

Fig. 1. Plots of u and v, where $\alpha = -0.005$, $k = -1$, and $r_1 = 0.5$.

Solution (32) is a bell-type solitary wave solutions for u and v. Solutions (33–37) are triangle-type periodical solutions (Solution (37) is depicted in Figure 1). Solution (38) is a singular bell-type solitary wave solution for u (i.e. that develops a singularity at a finite point) and a kink-type solitary wave solution for v, which are depicted in Figure 2. Solutions (39–45) are of triangle-type.

4. Conclusion

We have obtained new travelling wave solutions for the soliton breaking equation, the coupled KdV system. The obtained solutions include rational. periodical, singular and solitary wave solutions. Rational solutions may be helpful to explain certain physical phenomena. Because a rational solution is a disjoint union

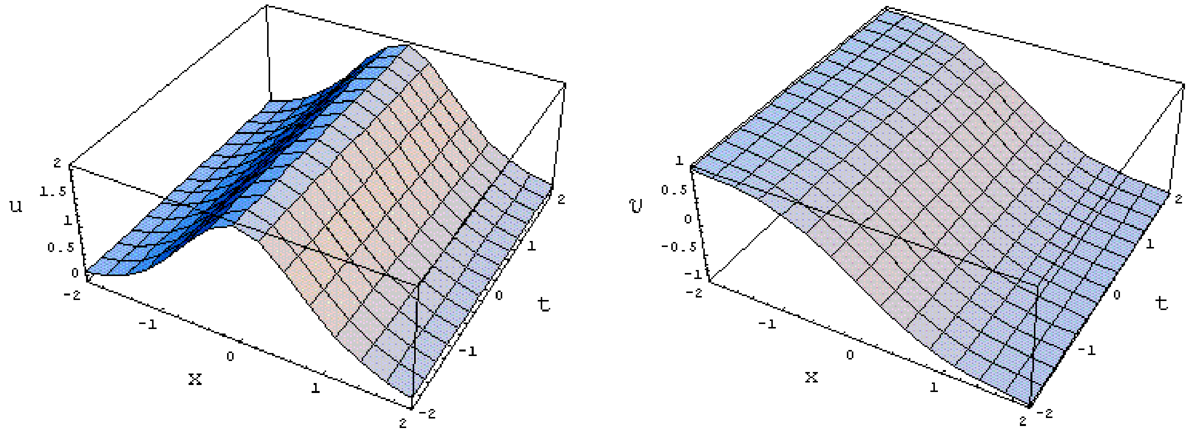

Fig. 2. Plots of u and v, where $\alpha = -0.005$, $k = -1$, and $s_1 = 0.5$.

of manifolds, particle systems describing the motion of a pole of rational solutions for a KdV equation were analyzed in [5, 11 – 13]. Triangle-type periodical solutions develop singularity at a finite point, i.e. for any fixed $t = t_0$ there exist an x_0 at which these solutions blow up (as shown in Fig. 1). There is much current interest in the formation of so called "hot-spots" or "blow-up" of solutions [5, 14 – 16]. It appears that the singular solutions will model these physical phenomena.

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